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UNIFORM EMBEDDINGS OF BANACH SPACES†

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ABSTRACT
It is proved that for $1 \leqq p \leqq 2$, $L_p(0,1)$ and l_p are uniformly equivalent to bounded subsets of themselves. It is also shown that for $1 \leqq p \leqq 2$, $1 \leqq q < \infty$, L_p is uniformly equivalent to a subset of l_q.

Two metric spaces X, Y are called uniformly equivalent, if there is a map T from X onto Y, such that T and T^{-1} are uniformly continuous. Very little is known about the theory of the uniform topology of Banach spaces. For example, it is still an open problem whether two uniformly equivalent Banach spaces are isomorphic‡ ([2] p. 283, [6] p. 1, and [8] p. 284). A problem of another type deals with embeddings. In [7] Gorin raised the problem whether l_2 is uniformly equivalent to a bounded subset of itself (see also [3] p. 48). This problem can naturally be asked for a general Banach space. In [1] it was proved that every separable Banach space containing c_0 is uniformly equivalent to a bounded subset of itself. Here we prove that for $1 \leqq p \leqq 2$, $L_p(0,1)$ (and l_p) is uniformly equivalent to a bounded subset of itself. This provides, in particular, an affirmative answer to the question of Gorin, mentioned above. An interesting consequence of our theorem is that for $1 \leqq p \leqq 2$, $1 \leqq q < \infty$, $L_p(0,1)$ is uniformly equivalent to a subset of l_q. This answers affirmatively a part of a problem raised by Enflo [6] p. 2.

Our main result is the following theorem:

THEOREM 1. *There is a map* $T: L_1(0,1) \to L_1(0,1)$ *satisfying*:

(a) *For every* $f \in L_1$, $\| Tf \| = 1$.

(b) *For every* $f, g \in L_1$ *with* $\| f - g \| \leqq 1$, *we have*

$$\| f - g \| / 3 \leqq \| Tf - Tg \| \leqq 2 \| f - g \|.$$

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‡See *Added in proof.*

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(c) *For every* $f, g \in L_1$ *with* $\|f - g\| \geqq 1$, *we have* $\|Tf - Tg\| \geqq \frac{1}{2}$.

For the proof of Theorem 1, it is more convenient to work with non-negative functions. We state, therefore, the following trivial proposition.

PROPOSITION 2. *There is an isometric embedding* U *of* $L_1(0,1)$ *into* $L_1^+(0,1) = \{f \in L_1 \mid f \geqq 0\}$.

We shall need the following.

LEMMA 3. *There is a dense subset* M *of* $L_1^+(0,1)$, *and a map* $V\colon M \to P(R)$ (= *The set of all subsets of the reals*) *so that*:

(a) *For every* $f \in M$, $V(f)$ *is measurable, and* $\mu(V(f)) = e^{\|f\|}$ (*where* μ *is the usual Lebesgue measure, and* $\|f\|$ *is the* L_1-*norm of* f).

(b) *For every* $f, g \in M$, $f \wedge g = \min\{f, g\} \in M$ *and* $V(f \wedge g) = V(f) \cap V(g)$.

We show first how to derive Theorem 1 from Proposition 2 and Lemma 3. We shall give, later on, the proofs of Proposition 2 and Lemma 3.

PROOF OF THEOREM 1. We define a map H from $L_1^+(0,1)$ into $L_1(R)$, and prove that H satisfies conditions (a), (b) and (c) of the theorem. Then, we use the isometric imbedding U defined by Proposition 2, and the known fact that there is an isometry G of $L_1(R)$ onto $L_1(0,1)$, and define $T\colon L_1(0,1) \to L_1(0,1)$ by $T = G \circ H \circ U$. T is the desired map.

Let M be the dense subset of $L_1^+(0,1)$ and let V be the map defined by Lemma 3. We define a map $H\colon M \to L_1(R)$ by

$$H(f) = \frac{\chi_{V(f)}}{\|\chi_{V(f)}\|} = \frac{\chi_{V(f)}}{e^{\|f\|}}$$

where χ_A denotes the characteristic function of the set A. From the definition, it follows that H satisfies the following:

(1)
(a) For every $f \in M$, $H(f) \geqq 0$.

(b) For every $f \in M$, $\|H(f)\| = 1$

(c) For every $f, g \in M$

$$Hf \wedge Hg = \frac{\chi_{V(f) \cap V(g)}}{\max\{e^{\|f\|}, e^{\|g\|}\}} = \frac{\chi_{V(f \wedge g)}}{e^{\max\{\|f\|, \|g\|\}}}. \tag{2}$$

Now take $f, g \in M$ with $\|f - g\| \leqq 1$. We assume without loss of generality that $\|f\| \geqq \|g\|$.

We have:

$$\|f-g\| = \|f\| + \|g\| - 2\|f \wedge g\| \leqq 2(\|f\| - \|f \wedge g\|) \tag{3}$$

hence

$$\tfrac{1}{2}\|f-g\| \leqq \|f\| - \|f \wedge g\| \leqq \|f-g\| \leqq 1. \tag{4}$$

On the other hand

$$\|Hf \wedge Hg\| = \frac{\|\chi_{V(f \wedge g)}\|}{e^{\|f\|}} = \frac{e^{\|f \wedge g\|}}{e^{\|f\|}} = e^{\|f \wedge g\| - \|f\|},$$

hence

$$\|Hf - Hg\| = \|Hf\| + \|Hg\| - 2\|Hf \wedge Hg\| = 2 - 2e^{\|f \wedge g\| - \|f\|}$$
$$= 2(1 - e^{-(\|f\| - \|f \wedge g\|)}).$$

But for every $0 \leqq x \leqq 1$ we have

$$\frac{x}{3} \leqq 1 - e^{-x} \leqq x$$

therefore

$$\tfrac{2}{3}(\|f\| - \|f \wedge g\|) \leqq \|Hf - Hg\| \leqq 2(\|f\| - \|f \wedge g\|) \tag{5}$$

which by (4) implies

$$\tfrac{1}{3}\|f-g\| \leqq \|Hf - Hg\| \leqq 2\|f-g\|. \tag{6}$$

If we take $f, g \in M$ with $\|f-g\| \geqq 1$ and $\|f\| \geqq \|g\|$, then by (3)

$$\|f\| - \|f \wedge g\| \geqq \|f-g\|/2 \geqq \tfrac{1}{2}.$$

Hence

$$\|Hf - Hg\| = 2(1 - e^{-(\|f\| - \|f \wedge g\|)}) \geqq 2(1 - e^{-\frac{1}{2}}) \geqq \tfrac{1}{2}. \tag{7}$$

By (1), (6) and (7), H satisfies the desired conditions on M. Now, we extend H by continuity to all $L_1^+(0,1)$. This concludes the proof of the theorem.

We prove now the proposition and the lemma.

Proof of Proposition 2. For every $f \in L_1(0,1)$ define:

$$f^+ = \max\{f, 0\}, \qquad f^- = \max\{-f, 0\}.$$

Define now $U: L_1(0,1) \to L_1^+(0,1)$ by

$$U(f)(t) = \begin{cases} 2f^+(2t) & 0 \leqq t \leqq \frac{1}{2} \\ 2f^-(2t-1) & \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

It is easy to see that U is an isometric imbedding.

PROOF OF LEMMA 3. We begin with some notations. Let $a > 0$ be a real number, and let $S^{(a)}: R^2 \to R$ be a one to one map satisfying

(1) $S^{(a)}$ and $S^{(a)-1}$ are measurable, measure-preserving maps.

(2) For every natural number m

$$S^{(a)}([0, e^{ma}] \times [0, e^{ma}]) = [0, e^{2ma}].$$

(The existence of such a map is an obvious and well known fact.)

For every $n \geqq 1$, define a map $S_n: R^{2^n} \to R^{2^{n-1}}$ by

$$S_n = \mathop{\times}_{i=1}^{2^{n-1}} S^{(2^{-(n-1)})}.$$

S_n is a one to one map satisfying:

(1) S_n and S_n^{-1} are measurable, measure-preserving maps.

(2) For every natural number m

$$S_n\left(\mathop{\times}_{i=n}^{2^{n-1}} [0, e^{m\, 2^{-(n-1)}}] \times [0, e^{m\, 2^{-(n-1)}}] \right) = \mathop{\times}_{i=1}^{2^{n-1}} [0, e^{2m \cdot 2^{-(n-1)}}].$$

Define now $T_n: R^{2^n} \to R$ by $T_n = S_1 \circ S_2 \circ \cdots S_n$.

For every n let Z_n be the following subset of $L_1(0,1)$:

$$Z_n = \left\{ f \in L_1 \mid f = \sum_{k=1}^{2^n} a_k \chi_{[(k-1)/2^n, k/2^n]}, \quad a_k = \frac{b_k}{2^n}, \quad b_k \text{ integers.} \right\}$$

and define

$$W_n: Z_n^+ \to P(R^{2^n})$$

by

$$W_n(f) = W_n\left(\sum_{k=1}^{2^n} a_k \chi_{[(k-1)/2^n, k/2^n]} \right) = \mathop{\times}_{k=1}^{2^n} [0, e^{a_k/2^n}].$$

W_n satisfies the following conditions:

(i) For every $f \in Z_n^+$, $W_n(f)$ is a measurable set and $\mu(W_n(f)) = e^{\|f\|}$.

(ii) For every $f, g \in Z_n^+$, $W_n(f \wedge g) = W_n(f) \cap W_n(g)$.

(iii) Let $I_n: Z_{n-1} \to Z_n$ be the canonical embedding.

Then the diagram

$$\begin{array}{ccc} Z_{n-1}^+ & \xrightarrow{W_{n-1}} & P(R^{2^{n-1}}) \\ \downarrow{\scriptstyle I_n} & & \uparrow{\scriptstyle S_n} \\ Z_n^+ & \xrightarrow{W_n} & P(R^{2^n}) \end{array}$$

is commutative. That is to say $W_{n-1} = S_n W_n I_n$. We define now the set $M = \bigcup_{n=1}^{\infty} Z_n^+$, and a map $V: M \to P(R)$. For every $f \in M$, there is n such that $f \in Z_n^+$. Define $V(f) = T_n W_n f$.

Condition (iii) above ensures that V is well defined. Clearly M is a dense subset of $L_1^+(0,1)$, and by conditions (i) and (ii) above, V satisfies the conditions required by the lemma. This proves the lemma.

For $1 \leqq p \leqq 2$, $L_p(0,1)$ is isometric to a subspace of $L_1(0,1)$ ([9] p. 139). By a theorem of Mazur [11], the unit ball of $L_1(0,1)$ is uniformly equivalent to the unit ball of $L_p(0,1)$, and to the unit ball of l_p, $1 \leqq p < \infty$. Thus we have:

THEOREM 4. *For $1 \leqq p \leqq 2$, $L_p(0,1)$ (respectively l_p) is uniformly equivalent to a bounded subset of itself.*

For $p = 2$ this answers affirmatively a problem raised by Gorin [7] (see also [3] p. 48). We do not know if this theorem remains true also for $2 < p < \infty$. For general Banach spaces, it is known that every separable infinite dimensional $C(K)$ space is uniformly equivalent to a bounded subset of itself [1]. (Indeed, this is true for every separable Banach space containing c_0.) We do not know if there exists an infinite dimensional Banach space, which is not uniformly equivalent to a bounded subset of itself.

Another consequence of Theorem 1 is the following:

THEOREM 5. *For $1 \leqq p \leqq 2$, $1 \leqq q < \infty$, $L_p(0,1)$ is uniformly equivalent to a subset of l_q (and therefore to a subset of $L_q(0,1)$).*

This theorem answers a part of a problem raised by Enflo ([6] p. 2). We do not know if this holds also for $2 < p < \infty$.

In connection with Theorem 5, it is of interest also to recall the following facts. By a theorem of Mankiewicz [10], $L_p(0,1)$ is not Lipschitz equivalent to a subset of l_q. (A metric space X is Lipschitz equivalent to a metric space Y, if there is a map T from X onto Y such that T and T^{-1} satisfy Lipschitz condition of first order.) Thus we have an example of Banach spaces X, Y such that X is uniformly equivalent to a subset of Y, but X is not Lipschitz equivalent to any subset of Y. Enflo [4] proved that a Banach space X which is uniformly equivalent to l_2, is isomorphic to l_2. (This result was recently sharpened by Ribe [12].) Theorem 5 shows the essential role played by the assumption in Enflo's result that the uniform homeomorphism maps X onto l_2. Another theorem of Enflo [5] states that c_0 is not uniformly equivalent to any subset of l_2. It would be interesting to characterize all Banach spaces which are uniformly equivalent to subsets of l_2.

Added in proof. Recently it was shown that there are two uniformly equivalent Banach spaces which are not isomorphic (I. Aharoni and J. Lindenstrauss, *Uniform equivalence between Banach spaces*, to appear).

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